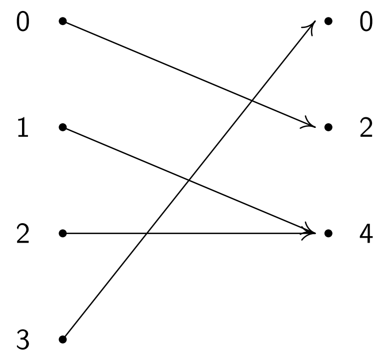
MATH 1650 FUNCTION ARITHMETIC

EXAMPLE: Let $f = \{(0,3),(1,2),(2,-5)\}$ and $g(t) = t - 7$. Find the following:

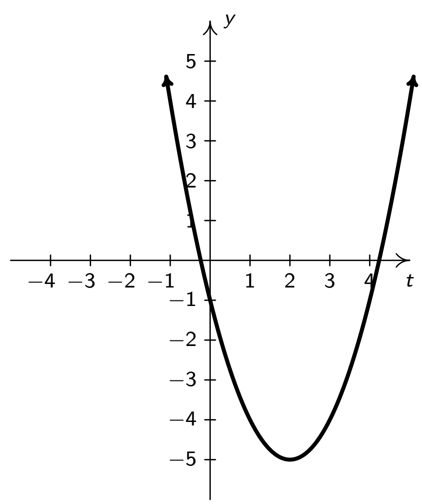
- $(f+g)(1) = f(1) + g(1) = 2 + (1-7) = -4$
- $(f-g)(0) = f(0) - g(0) = 3 - (0-7) = 10$
- $(fg)(0) = f(0)g(0) = (3)(0-7) = -21$
- $\left(\dfrac{g}{f}\right)(2) = \dfrac{g(2)}{f(2)} = \dfrac{2-7}{-5} = \dfrac{-5}{-5} = 1$
- $(g \circ f)(0) = g(f(0)) = g(3) = 3 - 7 = -4$
- $(f \circ g)(7) = f(g(7)) = f(7-7) = f(0) = 3$

EXAMPLE: Suppose f and g are described below, find the following:

The function f described below:



The function g whose graph is below:



- $(f+g)(2) = f(2) + g(2) = 4 + (-5) = -1$
- $(fg)(0) = f(0)g(0) = (2)(-1) = -2$
- $(g \circ f)(0) = g(f(0)) = g(2) = -5$
- $(f \circ f)(3) = f(f(3)) = f(0) = 2$

EXAMPLE: Let $f(x) = 3x - x^2$. Find and simplify $\dfrac{f(x+h) - f(x)}{h}$.

$f(x+h) = 3(x+h) - (x+h)^2 = 3x + 3h - \left(x^2 + 2xh + h^2\right) = 3x + 3h - x^2 - 2xh - h^2$. Hence:

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\left(3x + 3h - x^2 - 2xh - h^2\right) - \left(3x - x^2\right)}{h} \\
&= \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h} \\
&= \frac{3h - 2xh - h^2}{h} \\
&= \frac{\cancel{h}(3 - 2x - h)}{\cancel{h}} \\
\frac{f(x+h)-f(x)}{h} &= 3 - 2x - h
\end{aligned}$$

EXAMPLE: Find and simplify expressions for $(g \circ f)(x)$ and $(f \circ g)(x)$. List the domains for $g \circ f$ and $f \circ g$.

- $f(x) = x^2$, $g(x) = \sqrt{x}$

  $(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = \sqrt{x^2} = |x|.$

  To find the domain we analyze the formula before we simplify: $\sqrt{x^2}$

  Since x is squared before taking the square root, the domain of $g \circ f$ is $(-\infty, \infty)$.

  $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \left(\sqrt{x}\right)^2 = x.$

  To find the domain we analyze the formula before we simplify: $\left(\sqrt{x}\right)^2$.

  Since the square root of x is taken before it is squared, we need $x \geq 0$ so the domain of $f \circ g$ is $[0, \infty)$.

- $f(x) = \dfrac{2x}{x-1}$ and $g(x) = \dfrac{4}{x+1}$

  $$(g \circ f)(x) = g(f(x)) = g\left(\frac{2x}{x-1}\right) = \frac{4}{\dfrac{2x}{x-1}+1} = \frac{4}{\dfrac{2x}{x-1}+1} \cdot \frac{x-1}{x-1} = \frac{4(x-1)}{2x+1(x-1)} = \frac{4x-4}{3x-1}$$

  So $(g \circ f)(x) = \dfrac{4x-4}{3x-1}$. For domain, we look to the formula: $\dfrac{4}{\dfrac{2x}{x-1}+1}$.

  We have two denominators present so we solve each equal to zero to find excluded values.

  Solving $x - 1 = 0$ we get $x = 1$ is an excluded value.

  Solving $\dfrac{2x}{x-1} + 1 = 0$, we get $\dfrac{2x}{x-1} = -1$ so $2x = (-1)(x-1)$. This gives $2x = -x + 1$ or $3x = 1$.

  Hence, we have $x = \dfrac{1}{3}$ as another value excluded from the domain.

  So the domain of $g \circ f$ is $\left\{x \,|\, x \neq \dfrac{1}{3}, 1\right\}$, or $\left(-\infty, \dfrac{1}{3}\right) \cup \left(\dfrac{1}{3}, 1\right) \cup (1, \infty)$.

  $$(f \circ g)(x) = f(g(x)) = f\left(\frac{4}{x+1}\right) = \frac{2\left(\dfrac{4}{x+1}\right)}{\dfrac{4}{x+1}-1} = \frac{\dfrac{8}{x+1}}{\dfrac{4}{x+1}-1} \cdot \frac{x+1}{x+1} = \frac{8}{4-(x+1)} = \frac{8}{3-x}$$

  So $(f \circ g)(x) = \dfrac{8}{3-x}$. For domain, we look to the formula: $\dfrac{2\left(\dfrac{4}{x+1}\right)}{\dfrac{4}{x+1}-1}$.

  Once again we look for denominators. We have $x + 1 = 0$ when $x = -1$ so $x = -1$ is an excluded value.

  Solving $\dfrac{4}{x+1} - 1 = 0$ gives $\dfrac{4}{x+1} = 1$ or $4 = x + 1$. We get $x = 3$ as another excluded value.

  So the domain of $f \circ g$ is $\{x \,|\, x \neq -1, 3\}$, or $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

EXAMPLE: Let $h(x) = 4x - x^2$. Find functions f and g so that $h = f - g$.

Since $(f - g)(x) = f(x) - g(x)$, we let $f(x) = 4x$ and $g(x) = x^2$. Then $(f - g)(x) = f(x) - g(x) = 4x - x^2$.

EXAMPLE: Let $h(z) = \sqrt{4 - z}$. Find functions f and g so that $h = g \circ f$.

Since $(g \circ f)(z) = g(f(z))$ we let $f(z)$ be the 'inside' function, $f(z) = 4 - z$.

Then $g(z)$ is the 'outside' function $g(z) = \sqrt{z}$. Then $(g \circ f)(z) = g(f(z)) = g(4 - z) = \sqrt{4 - z}$.

EXAMPLE: Let $r(t) = \dfrac{3 - t}{t^2 + 1}$. Find functions f and g so that $r = \dfrac{f}{g}$.

Since $\left(\dfrac{f}{g}\right)(t) = \dfrac{f(t)}{g(t)}$, we let $f(t) = 3 - t$ and $g(t) = t^2 + 1$. Then, $\left(\dfrac{f}{g}\right)(t) = \dfrac{f(t)}{g(t)} = \dfrac{3 - t}{t^2 + 1}$.

EXAMPLE: Let $R(t) = \sqrt{\dfrac{3 - t}{t^2 + 1}}$. Find functions f, g, and h so that $R = h \circ \left(\dfrac{f}{g}\right)$.

Since $\left(h \circ \left(\dfrac{f}{g}\right)\right)(t) = h\left(\dfrac{f}{g}(t)\right) = h\left(\dfrac{f(t)}{g(t)}\right)$, we let $\dfrac{f(t)}{g(t)} = \dfrac{3 - t}{t^2 + 1}$ and $h(t) = \sqrt{t}$.

Hence, $f(t) = 3 - t$, $g(t) = t^2 + 1$, and $h(t) = \sqrt{t}$. We check:

$$\left(h \circ \left(\frac{f}{g}\right)\right)(t) = h\left(\frac{f}{g}(t)\right) = h\left(\frac{f(t)}{g(t)}\right) = h\left(\frac{3 - t}{t^2 + 1}\right) = \sqrt{\frac{3 - t}{t^2 + 1}}$$